7.1 SUMS OF RANDOM VARIABLES

We have talked about how to take expectations of sums.

$$E[X_1 + X_2 + \cdots + X_n]$$

Then we talked about the variance of sums

$$\mathrm{Var}(X_1 + \cdots + X_n) = \mathrm{Var}(X_1) + \cdots + \mathrm{Var}(X_n) + \sum_{i \neq j} \mathrm{Cov}(X_i, X_j)$$

When the random variables are independent

$$\sum_{i,j} \mathrm{Cov}(X_i, X_j) = 0$$

and the formula for variance simplifies.

Then we talked about the mgf's of independent sums and how they become products

$$E\left[e^{t(X_1 + X_2 + \cdots + X_n)}\right] = E\left[e^{tX_1}\right] \cdots E\left[e^{tX_n}\right]$$

Next we want to talk about the whole distribution of sums (their pmf and their cdf)

$X_1 + X_2 + \cdots + X_n$

Expected value ✓

Variance ✓

MGF ✓

are the same as regular of integrals.

CONVOLUTION OF RVS

MGFs.

7.1 Ex: Suppose $X \sim$ Poisson(λ), $Y \sim$ Poisson(μ). Find distribution/pmf of $X+Y$ if X, Y independent.

Guess: Since X and Y represent arrivals of customers in one time period, then if X, Y are independent $X+Y \sim$ Poisson$(\lambda+\mu)$.

$$e^{(\lambda+\mu)(e^t-1)} = e^{\lambda(e^t-1)} e^{\mu(e^t-1)}$$

$\Rightarrow$ Poisson$(\lambda+\mu)$

Let's try

$$P(X+Y=0) = p_{X+Y}(0) = P(X=0, Y=0) \quad \text{independent}$$

$$= e^{-\lambda} \cdot e^{-\mu} = e^{-(\lambda+\mu)}$$

$P(X=0) = e^{-\lambda}$

$P(Y = e^{-\mu})$

$P(X=k) = \dfrac{e^{-\lambda}\lambda^k}{k!}$.

$$p_{X+Y}(1) = P(X=1, Y=0) + P(X=0, Y=1)$$

$$= \frac{e^{-\lambda}\lambda^1}{1!} e^{-\mu} + \frac{e^{-\mu}\mu^1}{1!} e^{-\lambda}$$

$$= \frac{e^{-(\lambda+\mu)}(\lambda+\mu)^1}{1!}$$

Using this idea.

In general

$$p_{X+Y}(k) = \sum_{i=0}^{k} P(X=i, Y=k-i)$$

breaking up the event into various possibilities law of total probability.

$$p_{X+Y}(2) = \underbrace{P(X=0, Y=2)}_{i=0} + \underbrace{P(X=1, Y=1)}_{i=1} + \underbrace{P(X=2, Y=0)}_{i=2}$$

Check this on your own.

$$p_{X+Y}(k) = \sum_{i=0}^{k} \frac{e^{-\lambda}\lambda^{i}}{i!} \frac{e^{-\mu}\mu^{(k-i)}}{(k-i)!}$$

$$= \frac{e^{-(\lambda+\mu)}}{n!} \sum_{k=0}^{n} \frac{n!}{k!(n-k)!} \lambda^{k} \mu^{n-k}$$

$$\sim \text{Poisson}(\lambda+\mu)$$

So lets generalize this: For <u>any</u> two rvs taking integer values

$$p_{X+Y}(n) = \sum_i \mathbb{P}(X=i, Y=n-i)$$

joint pmf.

$$= \sum_i \underbrace{p_{XY}(i, n-i)}_{\text{Joint probability}}$$

If X and Y are independent

then

becomes a product of the indiv. pmf.

$$p_{XY}(i, n-i) = p_X(i)\, p_Y(n-i)$$

If X, Y are two independent rvs.

(discrete) $$p_{X+Y}(n) = \sum_j p_X(j)\, p_Y(n-j)$$

Similarly, for continuous:

(continuous) $$f_{X+Y}(t) = \int_{-\infty}^{\infty} f_X(u)\, f_Y(t-u)\, du$$

$X \sim \text{Uniform}(0,1) \qquad Y \sim \text{Exp}(3).$

$Z = X + Y$. Find the pdf of Z.

i) Using areas:

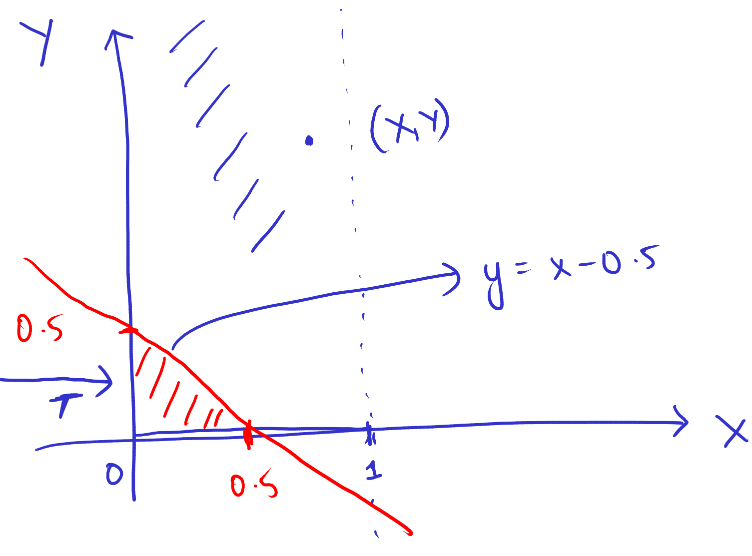
Find the cdf of Z: $F_Z(t) = \underbrace{P(Z \leq t)}$

how do you compute this?

$$\frac{d}{dt} F_Z(t) = f_Z(t)$$

$P(X+Y \leq 0.5)$

$\frac{1}{2}(0.5)^2$



$$= \iint_T f_{X,Y}(u,v)\, du\, dv$$

$$= \int_0^{0.5} \int_0^{0.5-v} f_{X,Y}(u,v)\, du\, dv$$

$X \sim \text{Uniform}(0,1)$

$Y \sim \text{Exp}(3)$, X & Y are independent.

$$f_X(u) = \begin{cases} 1 & 0<u<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(v) = \begin{cases} 3e^{-3v} & v \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \int_0^{0.5}\int_0^{0.5-\vartheta} \underbrace{f_{X,Y}(u,\vartheta)}\, du\, d\vartheta$$

$$f_X(u) = \begin{cases} 1 & 0<u<1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(\vartheta) = \begin{cases} 3e^{-3\vartheta} & \vartheta \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$= \int_0^{0.5}\int_0^{0.5-\vartheta} 3e^{-3\vartheta}\, du\, d\vartheta$$

$$= \int_0^{0.5} (0.5-\vartheta)\, 3e^{-3\vartheta}\, d\vartheta \quad = -3(0.5)\frac{e^{-3\vartheta}}{3}\Big|_0^{0.5} = (0.5)(1-e^{-3(0.5)})$$

$$\int_0^{0.5} \vartheta e^{-3\vartheta}\, d\vartheta = \frac{\vartheta e^{-3\vartheta}}{-3}\Big|_0^{0.5} + \int_0^{0.5} e^{-3\vartheta}\, d\vartheta = \frac{0.5e^{-3(0.5)}}{-3} + \frac{e^{-3\vartheta}}{-3}\Big|_0^{0.5}$$

$$= -\frac{0.5}{3}e^{-3(0.5)} + \frac{1-e^{-3(0.5)}}{3}$$

$$P(X+Y \leq 0.5) = (0.5)(1-e^{-3(0.5)}) + 0.5e^{-3(0.5)} - (1-e^{-3(0.5)})$$

$$P(X+Y \leq t)$$

Suppose X is a $\mathrm{Exp}(3)$ random variable and Y is a $\mathrm{Unif}\,[0,1]$ r.v.
Find the pdf of $Z = X+Y$.

Assume X and Y are independent

In your webwork you found

$$P(Z \leq a) = F_Z(a)$$

So to find the pdf of Z, you may simply use

$$f_Z(t) = \frac{d}{dt} F_Z(t)$$

Instead if we use convolution:

$$f_{X+Y}(t) = \int_{-\infty}^{\infty} f_{X,Y}(a, t-a) \cdot da$$

But $f_{X,Y}(a, t-a) = f_X(a)\, f_Y(t-a)$

since X and Y are indep.

$$f_X(a) = 3e^{-3a} \qquad a > 0.$$

$$f_Y(a-t) = \begin{cases} 1 & 0 \le t-a \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

$$t-1 \le a \le t$$

Let us 1st suppose that $0 < t < 1$

Then a varies between 0 and t

$$\Rightarrow f_{X+Y}(t) = \int_0^t 3e^{-3a} \, 1 \, da$$

$$= 1 - e^{-3t} \qquad 0 < t < 1$$

If $t > 1$ then

$$f_{X+Y}(t) = \int_{t-1}^{t} 3e^{-3a} \, da = e^{-3(t-1)} - e^{-3t}$$

Convolution of Normal random variables.

Let $X_1 \sim N(0, \sigma_1)$ and $X_2 \sim N(0, \sigma_2)$

What is the distribution of $X_1 + X_2$?

Use MGFs: we have seen that

$$E\left[e^{t(X_1+X_2)}\right] = E\left[e^{tX_1}\right] E\left[e^{tX_2}\right]$$

For Normal rvs, we have

$$E\left[e^{tX_i}\right] = e^{\frac{t^2 \sigma_i^2}{2}}$$

So

$$E\left[e^{t(X_1+X_2)}\right] =$$

POLL

What is the distribution of $X_1 + X_2$?

Let us do this using convolution:

$$p_{X_1+X_2}(t) = \int_{-\infty}^{\infty} \frac{e^{-\frac{x^2}{2\sigma_1}}}{\sqrt{2\pi\sigma_1}} \frac{e^{-\frac{(t-x)^2}{2\sigma_2}}}{\sqrt{2\pi\sigma_2}} \, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma_1\sigma_2}} \int_{-\infty}^{\infty} \exp\left(\frac{\sigma_2 x^2 + \sigma_1 (t-x)^2}{2\sigma_1\sigma_2} \right) dx$$

$$\frac{\sigma_2 x^2 + \sigma_1 (t-x)^2}{2\sigma_1\sigma_2} = \frac{x^2(\sigma_1+\sigma_2) - 2\sigma_1 t x + \sigma_1 t^2}{2\sigma_1\sigma_2}$$

$$= \frac{\left(\sqrt{\sigma_1+\sigma_2}\, x - \frac{\sigma_1 t}{\sqrt{\sigma_1+\sigma_2}}\right)^2 + \sigma_1 t^2 - \frac{\sigma_1^2 t^2}{\sigma_1+\sigma_2}}{2\sigma_1\sigma_2}$$

Make a COV $\quad u = \left(\sqrt{\sigma_1+\sigma_2}\, x - \frac{\sqrt{\sigma_1}\, t}{\sqrt{\sigma_1+\sigma_2}}\right)$

$$du = \sqrt{\sigma_1+\sigma_2}\, dx$$

$$= \frac{1}{\sqrt{2\pi\sigma_1\sigma_2}} \int_{-\infty}^{\infty} \exp\left(\frac{-u^2}{2\sigma_1\sigma_2}\right) \exp\left(\frac{-t^2}{2\sigma_1\sigma_2}\left(\frac{\sigma_1\sigma_2}{\sigma_1+\sigma_2}\right)\right) \frac{du}{\sqrt{\sigma_1+\sigma_2}}$$

$$= \frac{1}{\sqrt{2\pi(\sigma_1+\sigma_2)}} \exp\left(-\frac{t^2}{2(\sigma_1+\sigma_2)}\right)$$

$$N(\mu_1, \sigma_1) \quad N(\mu_2, \sigma_2) \stackrel{d}{=} N(\mu_1+\mu_2, \sigma_1+\sigma_2).$$

This is an IMPORTANT computation.

The sum of two independent normal random variables is also normal.

Ex: Red Sox and Yankees play a best of 7 series. What is the probability that the red sox will win in exactly 6 games?

This is a good gambling question.

Based on historical record

$p = P(\text{Red Sox win}) = 0.53.$

So as soon as one of the teams gets to 4 games, the series ends.

Example scores:

4-0
4-1
4-2
4-3

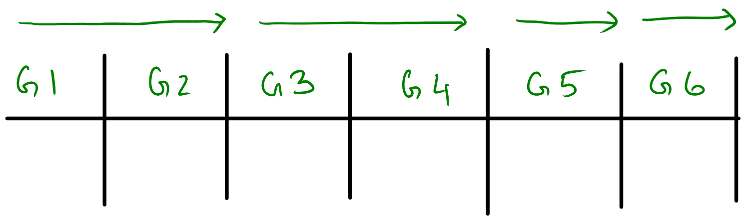
So if the Red Sox win in exactly 6 games, then the score must be 4-2, and the Red Sox MUST win game 6, to end the series.

let X_1 = time of 1st win

X_2 = time between 1st and 2nd win (bit imprecise)

X_3 = " " "

X_4 = " " "



Q: Are X_1, X_2, X_3, X_4 independent?

Q: Are they identically distributed?

Q: What is their distribution?

$$P(X_1 + X_2 + X_3 + X_4 = 6) = \sum_{1 \leq a_i \leq} P(X_1 = a_1, \dots, X_4 = a_4)$$

Let us reinterpret this: suppose the score is 4-2 at the end. Then

G1	G2	G3	G4	G5	G6

$$P(X_1 + X_2 + X_3 + X_4 = 6) = \binom{5}{3} p^3 (1-p)^2$$

Sum of 4 geometrics is called negative binomial distribution. We have computed the probability that a negative binomial random variable takes the value 6.

A negative binomial counts the time at which you get your n^{th} success, given that your chance of success is p:

Let $X \sim \text{Neg.Bin}(n, p)$

<u>POLL</u>

What is the range of X?

A

$\{n, n+1\}$

B

$\{0, 1, 2, \ldots, n\}$

C

$\{n, n+1, \ldots\}$

$$P(X=k) = \overbrace{\underset{1}{\cdot} \quad \underset{2}{\cdot} \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot}^{k-1 \text{ successes distributed}} \quad \underset{k}{\odot} \rightarrow \text{time of } n^{th} \text{ success.}$$

$$= \binom{k-1}{n-1} p^{n-1} (1-p)^{k-n}$$